

Closing Thu: 12.4(1)(2), 12.5(1)
 Closing next Tue: 12.5(2)(3), 12.6
 Closing next Thu: 13.1, 13.2

12.5 Lines and Planes in 3D

Lines: We use parametric equations for 3D lines. Here's a 2D warm-up:

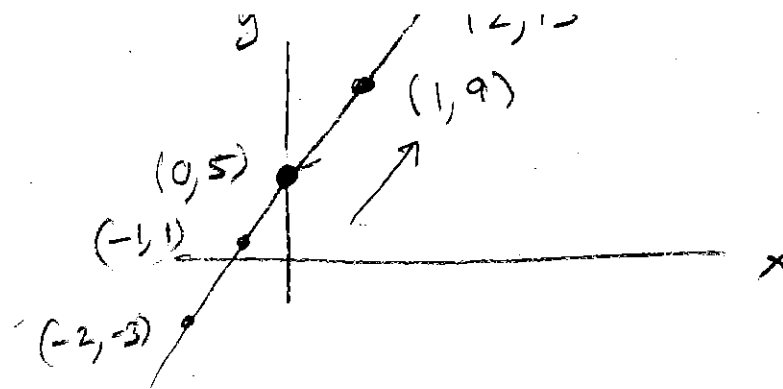
Consider the line: $y = 4x + 5$.

(a) Find a vector parallel to the line.

Call it \mathbf{v} .

(b) Find a vector whose head touches the line when drawn from the origin. Call it \mathbf{r}_0 .

(c) Observe, we can reach all other points on the line by walking along \mathbf{r}_0 , then adding scale multiples of \mathbf{v} .



$$\mathbf{v} = \langle 1, 4 \rangle \quad (\text{or } \langle 2, 8 \rangle \text{ or } \langle 3, 12 \rangle \text{ etc...})$$

$$\mathbf{r}_0 = \langle 0, 5 \rangle \quad (\text{or } \langle -1, 1 \rangle \text{ or } \langle 1, 9 \rangle \text{ etc...})$$

Let $(x, y) =$ ANOTHER POINT ON THE LINE

THEN \mathbf{v} ↙ SOME CONSTANT MULTIPLE

$$\langle x, y \rangle = \langle 0, 5 \rangle + t \langle 1, 4 \rangle$$

$$\Rightarrow \underbrace{x = 0 + t, \quad y = 5 + 4t}_{\text{ONE PARAMETERIZATION}}$$

$$\begin{array}{l} \text{EX] } (1, 9) \Leftrightarrow t = 1 \\ (2, 13) \Leftrightarrow t = 2 \\ (-1, 1) \Leftrightarrow t = -1 \end{array} \left. \vphantom{\begin{array}{l} (1, 9) \\ (2, 13) \\ (-1, 1) \end{array}} \right\} \text{FOR THIS PARAMETERIZATION}$$

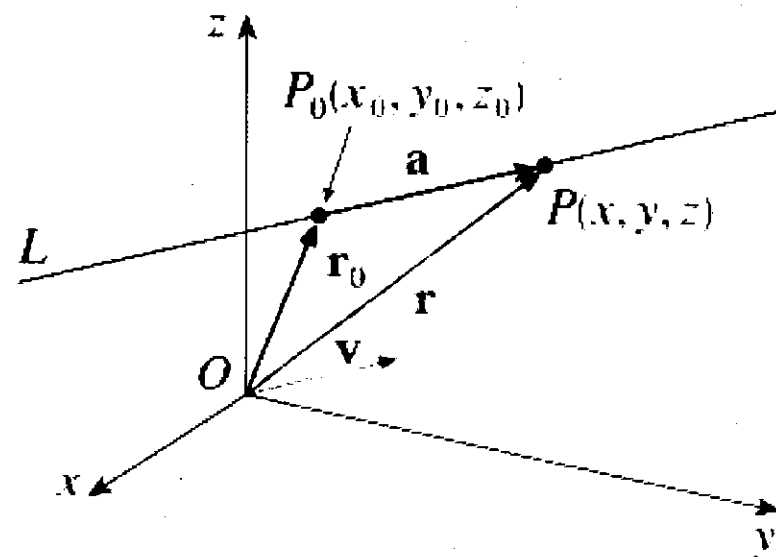
This same idea works to describe any line in 2- or 3-dimensions.

Summary of Line Equations

Let (x, y, z) be any point on the line and $\mathbf{r} = \langle x, y, z \rangle =$ a vector pointing to this point from the origin.

Find a direction vector and a point on the line.

1. $\mathbf{v} = \langle a, b, c \rangle$ *direction vector*
2. $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ *position vector*



$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

vector form

$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$$

parametric form

$$x = x_0 + at,$$

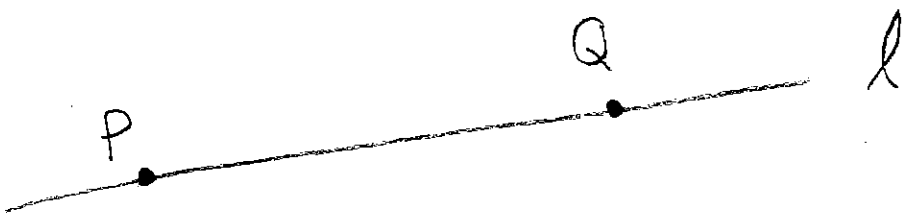
$$y = y_0 + bt,$$

$$z = z_0 + ct.$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

symmetric form

Basic Example – Given Two Points:
 Find parametric equations of the line
 thru $P(3, 0, 2)$ and $Q(-1, 2, 7)$.



NEED

1 A POINT: $\vec{r}_0 = \langle 3, 0, 2 \rangle$ (OR COULD USE $\langle -1, 2, 7 \rangle$)

2 A DIRECTION: $\vec{v} = \vec{PQ} = \langle -1-3, 2-0, 7-2 \rangle = \langle -4, 2, 5 \rangle$ (OR COULD USE \vec{QP})

$$\langle x, y, z \rangle = \langle 3, 0, 2 \rangle + t \langle -4, 2, 5 \rangle \quad \leftarrow \text{VECTOR FORM}$$

$$\begin{aligned} x &= 3 - 4t \\ y &= 0 + 2t \\ z &= 2 + 5t \end{aligned} \quad \leftarrow \text{PARAMETRIC FORM}$$

$$t = \frac{x-3}{-4}, \quad t = \frac{y-0}{2}, \quad t = \frac{z-2}{5}$$

$$\frac{x-3}{-4} = \frac{y-0}{2} = \frac{z-2}{5}$$

Symmetric form \uparrow

ASIDE)

IS $(11, -4, -2)$ ON THIS LINE?

$$\left. \begin{aligned} 11 &= 3 - 4t \Leftrightarrow t = -2 \\ -4 &= 0 + 2t \Leftrightarrow t = -2 \\ -2 &= 2 + 5t \Leftrightarrow t = 0 \end{aligned} \right\} \text{No!}$$

$(11, -4, -2)$ IS ON THE LINE

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

L1

$$\begin{aligned} x &= 4t \\ y &= 1 - t \\ z &= 7 + 2t \end{aligned}$$

Answer

$$\begin{aligned} x &= 1 + 4t \\ y &= 5 - t \\ z &= -2 + 2t \end{aligned}$$

EX) FIND THE EQUATION FOR THE LINE THRU $(1, 5, -2)$ AND PARALLEL TO L1

2. Two lines **intersect** if they have an (x, y, z) point in common.

EX)

L1

$$\begin{aligned} x &= t \\ y &= 1 + 2t \\ z &= 2 - 3t \end{aligned}$$

L2

$$\begin{aligned} x &= 3 - 4u \\ y &= 2 - 3u \\ z &= 1 + 2u \end{aligned}$$

Use different parameters when you combine!

$$\begin{aligned} \textcircled{i} \quad t &= 3 - 4u \\ \textcircled{ii} \quad 1 + 2t &= 2 - 3u \end{aligned} \quad \left. \begin{array}{l} ? \\ ? \end{array} \right\} \begin{aligned} 1 + 2(3 - 4u) &= 2 - 3u \\ \Rightarrow 1 + 6 - 8u &= 2 - 3u \\ \Rightarrow 5 &= 5u \quad \boxed{u=1} \\ &\Rightarrow \boxed{t=-1} \end{aligned}$$

L2

$$\begin{aligned} z + 3t &= 1 + 2u \\ -1 &\neq 3 \end{aligned}$$

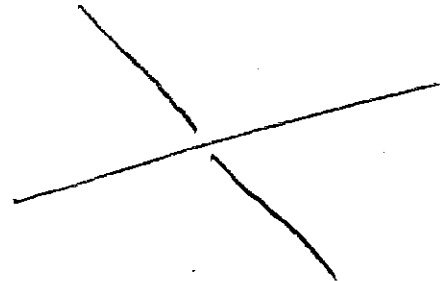
NO

DO NOT INTERSECT

Note: The *acute angle of intersection* is the acute angle between the direction vectors.

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| |\vec{v}_2| \cos \theta$$

3. Two lines are **skew** if they don't intersect and aren't parallel.

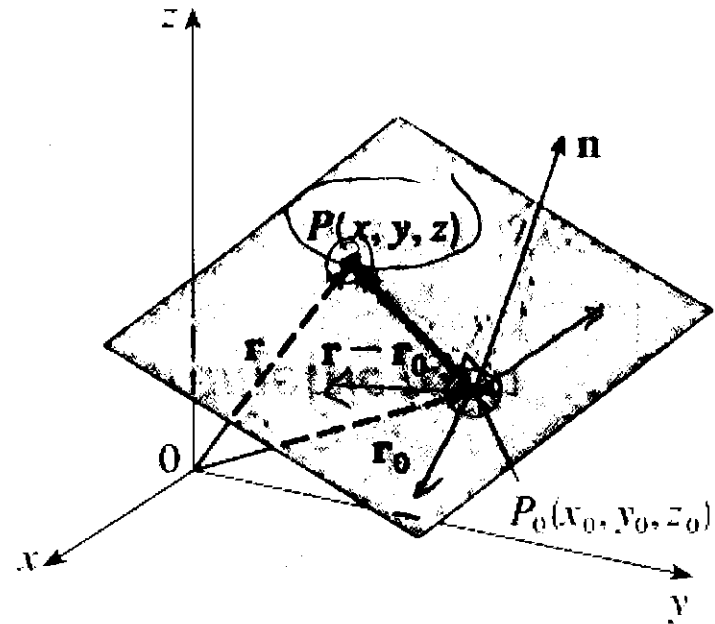


Summary of Plane Equations

Let (x,y,z) be any point on the plane and $\mathbf{r} = \langle x, y, z \rangle =$ a vector pointing to this point from the origin.

Find a normal vector and a point on the plane.

1. $\mathbf{n} = \langle a, b, c \rangle$ normal vector
2. $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ position vector



Normal Other PT Given points

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

vector form

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

standard form

If you expand out standard form you can write:

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

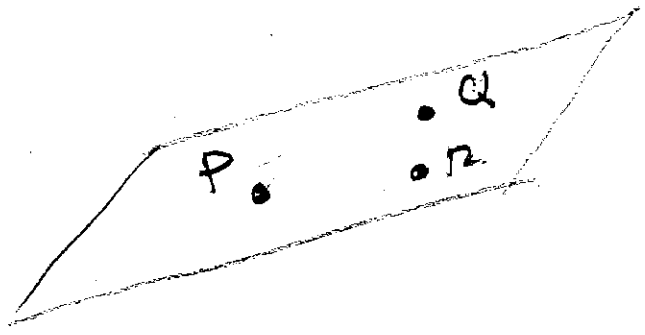
$$ax + by + cz = d \quad , \quad \text{where } d = ax_0 + by_0 + cz_0$$

Basic Example – Given Three Points:

Find the equation for the plane

through the points P(0, 1, 0),

Q(3, 1, 4), and R(-1, 0, 0)



NEED

1 A POINT : $\vec{r}_0 = \langle 0, 1, 0 \rangle$ (or $\langle 3, 1, 4 \rangle$ or $\langle -1, 0, 0 \rangle$)

2 A NORMAL : $\vec{PQ} = \langle 3, 0, 4 \rangle$, $\vec{PR} = \langle -1, -1, 0 \rangle$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 4 \\ -1 & -1 & 0 \end{vmatrix} = (0 - -4)\hat{i} - (0 - -4)\hat{j} + (-3 - -0)\hat{k}$$

$$= \langle 4, -4, -3 \rangle$$

CHECK!
12 + 0 - 12 = 0 ✓
-4 + 4 + 0 = 0 ✓

$$4(x-0) - 4(y-1) - 3(z-0) = 0$$

$$4x - 4y + 4 - 3z = 0$$

$$4x - 4y - 3z = -4$$

General Plane Facts

1. Two planes are **parallel** if their normal vectors are parallel.

EX GIVEN

$$3x + 4y + 10z = 14$$

FIND THE PLANE THAT IS PARALLEL AND PASSES THROUGH $(7, 15, -8)$

$$\text{ANSWER: } 3(x-7) + 4(y-15) + 10(z+8) = 0$$

2. If two planes are not parallel, then they must intersect to form a line.

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| |\vec{n}_2| \cos \theta$$

2a. The *acute angle of intersection* is the acute angle between their normal vectors.

2b. The planes are orthogonal if their normal vectors are orthogonal.

EX

$$\boxed{P1}: 2x + y - z = 10$$

$$\boxed{P2}: x - y + 3z = 2$$

FIND TWO POINTS OF INTERSECTION.

1] COMBINE $\Rightarrow 3x + 2z = 12$

2] YOU PICK POINTS!
 $\Rightarrow x=0 \Rightarrow z=6$
 $z=0 \Rightarrow x=4$

I PICKED THESE!

3] GO BACK TO GET y
 $x=0, z=6 \Rightarrow \begin{cases} 0+y-6=10 \\ 0-y+18=2 \end{cases} \Rightarrow y=16$

$$z=0, x=4 \Rightarrow \begin{cases} 8+y-0=10 \\ 4-y+0=2 \end{cases} \Rightarrow y=2$$

$(0, 16, 6)$
$(4, 2, 0)$

HERE ARE

TWO POINTS ON INTERSECTION

12.5 Summary

Lines: Find a POINT and DIRECTION.

$$\mathbf{v} = \langle a, b, c \rangle \quad \text{direction vector}$$

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \quad \text{position vector}$$

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$

Planes: Find a POINT and NORMAL

$$\mathbf{n} = \langle a, b, c \rangle \quad \text{normal vector}$$

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \quad \text{position vector}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

To find equations for a line

Info
given?

Find two points

Done.

$$\vec{v} = \overrightarrow{AB}$$

(subtract
components)

$$\vec{r}_0 = \vec{A}$$

lines parallel – directions parallel.

lines intersect – make (x,y,z) all equal
(different param!)

Otherwise, we say they are skew.

To find the equation for a plane

Info given?

Find three points

Done.

Two vectors parallel to the
plane: \overrightarrow{AB} and \overrightarrow{AC}

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\vec{r}_0 = \vec{A}$$

planes parallel – normals parallel.

Otherwise, the planes intersect.